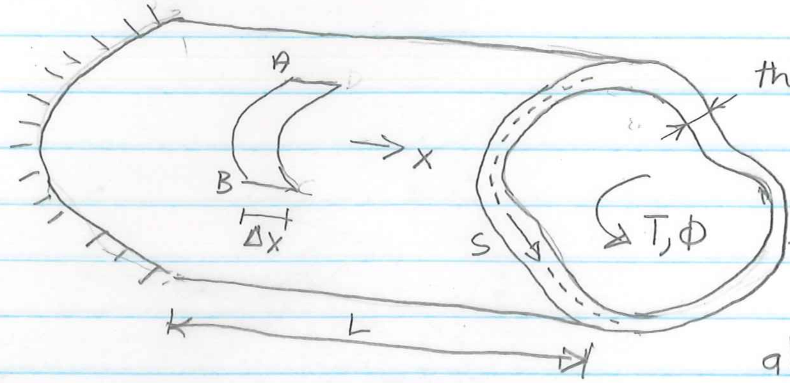
13. Twisting of a hollow shaft with thin wall (linearly elastic, homogeneous, isotropic material)

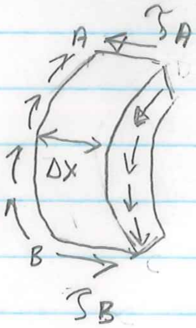


thickness of wall $= t$

t is a function of s (the coordinate along the mid thickness). t will also be assumed to be small.

The cross-section of the shaft does not change along the length.
Examine a free-body diagram of the piece 

cut through the thickness.



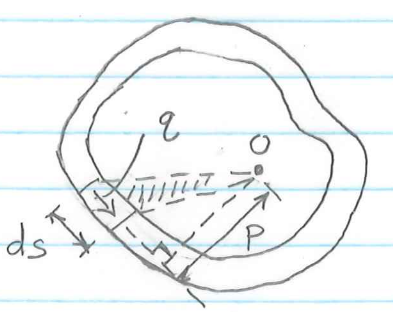
Sum forces in the x direction:

$$-\bar{\tau}_A t_A \Delta x + \bar{\tau}_B t_B \Delta x = 0$$

where the overbar denotes average through the thickness, t_A = thickness of wall at A,
t_B = thickness of wall at B

This is equivalent to $q(s) =$ constant
where $q(s) = \bar{\tau}(s) \cdot t(s)$ = the "shear flow".

The torque T can be determined in terms of q.



$$T = \sum M_o = \oint q\, p\, ds = q \oint p\, ds$$

Note that $p\, ds$ is twice the area of the shaded triangle. Therefore,

$T = 2q\mathcal{A}$ where $\mathcal{A}$ is the area enclosed by the mid-thickness line.

An expression for $\bar{\tau}(s)$ in terms of T can be obtained by substituting for q:

$$\bar{\tau}(s) = \frac{T}{2\,t(s)\,\mathcal{A}}$$

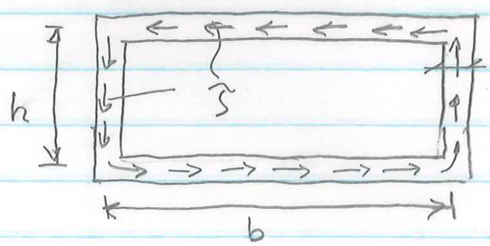
If $t(s)$ is small compared to the radius of curvature of the wall, then $\bar{\tau}(s)$ will be approximately constant through the wall thickness, so, $\bar{\tau}(s) = \tau(s)$.

The expression relating torque T to end rotation ϕ is most conveniently derived using work-energy concepts. This will be done later, but the expression is

$$\frac{T}{\phi} = \frac{JG}{L}$$

where $J = 4\mathcal{A}^2 \Big/ \oint \frac{ds}{t(s)}$

Example:



wall thickness t (constant)

We have

$$\tau = \frac{T}{2thb}, \qquad J = \frac{4h^2b^2}{2(b+h)/t} = \frac{2h^2b^2t}{b+h}$$

Note: If the shaft cross-section is anything but circular with constant thickness, sections will no longer remain planar as they rotate. This behavior is called warping. If warping is restrained at supports, then the twisting stiffness will be higher, and additional stresses (including axial stresses) will be induced in the shaft.